PARTICULAR AND GENERAL IN EARLY SYMBOLIC MANIPULATION

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The teaching and learning of early algebra draw heavily on arithmetic and the relationship between these two forms of activity is much debated. Drawing on interviews with 12 year old pupils we consider the ways in which some pupils used substitution of numbers for letters to extend their ability to manipulate algebraic expressions. We argue that teaching programmes need to emphasise 'seeing the particular in the general' alongside the manipulation of general expressions.

INTRODUCTION

Studies of the early stages of algebraic understanding have often been premised on the idea that algebra is a natural extension of arithmetic. This premise gives rise to a host of studies which have treated as crucial the move from arithmetic to algebraic thinking (e.g. Filloy and Rojano, 1984). Though the debate about where arithmetic ends and algebra begins has never quite been resolved, researchers have seemed to focus on finding where algebra becomes difficult, and how teaching interventions might be designed to enable pupils to overcome cognitive obstacles, bridge cognitive gaps or cross the didactic cut (Herscovics and Linchevski 1994, Filloy and Rojano 1989).

More recent work has argued that the late introduction of algebraic thinking in the school curriculum is at least in part responsible for pupils' subsequent difficulties (Carraher, Schliemann and Brizuela, 2001). Carraher et al describe how children aged 8 and 9 years began to use algebraic notation, though whether they were operating on the unknown was disputed by the research forum respondents (Linchevski, 2001, Tall, 2001). This study, and others looking at algebraic activity amongst young children (for example Blanton & Kaput, 2002; Slavitt, 1999), set the context for a widespread conviction that the teaching of arithmetic in elementary schools must be seen as providing the foundations from which algebraic thought can develop. This imperative is now enshrined in policy documents in both the US (NCTM, 2000) and the UK (DfEE, 1999).

Two kinds of activities which might be seen as drawing directly on pupils' arithmetic experience are the simplification of expressions through symbol manipulation (which tends to encourage 'letter as object' thinking) and the evaluation of expressions through substitution (which tend to encourage thinking of the letter as standing for a number). The ways in which algebra is typically introduced in schools tend to treat these two activities separately, rather than making explicit links between them. In this paper we look at some evidence of the way in which 12 year old pupils make use of their arithmetic understanding to work on problems which require manipulation of algebraic symbols. We make the assumption that when a child sees the usefulness of making a substitution of a particular value for an algebraic symbol, then this is one instance of what

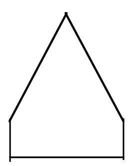
Mason (1993) calls ‘seeing the particular in the general’. It is a demonstration that the child sees the generality, and not just the symbol.

BACKGROUND

The data reported on here was collected as part of the Purposeful Algebraic Activity Project[1], a three year longitudinal study. In this study we are designing and using a series of algebraic tasks based on the use of spreadsheets with 11-13 year olds in two local comprehensive schools. One of the key features of these tasks is that they make use of the spreadsheet as both a context and a motive for generalising arithmetic processes (see Ainley, Bills and Wilson (forthcoming)).

As part of the project we are conducting interviews with two cohorts of pupils at different stages during the early years of secondary schooling. Through these interviews we intend to track pupils’ developing understanding and algebraic proficiency. The interviews on which this paper is based were conducted with cohort A, when they were aged 12, at the end of their first year in secondary school.

The interview questions were chosen to explore a variety of aspects of pupils’ algebraic capabilities. The three questions on which we report here were selected to examine pupils’ ability, at a simple level, to use the normal rules of arithmetic to simplify expressions that contain letters standing for numbers. They were:

Question 3:What is the distance around t	Question 12:	**Question 13: Are these statements true or false?**
	Simplify the following: i. $2a + 5a =$ ii. $2a + 5b + a =$ iii. $3a - b + a =$ iv. $(a - b) + b =$	i. $\frac{6+6+6}{3} = 6$ ii. $\frac{x+x+x}{3} = x$

Questions 3 and 12 are drawn from the CSMS survey (Küchemann, 1981). Correct answers to question 3 and the first two parts of question 12 could be given using only a ‘letter as object’ type of reasoning (Küchemann, 1981). Question 3 gave a context for the interpretation of the value of the letter, whereas question 12 gave none. Parts (iii) and (iv) of question 12 introduced negative signs or subtraction operations. Whereas in part (iii) the term $-b$ could be treated independently, and therefore as an object, in part (iv) an operation had to be performed on $-b$ in that it had to be combined with $+b$. We therefore expected the four parts of question 12 to have decreasing facility, as was found by Küchemann.

Question 13 was adapted from a question used in a national test set for 14 year olds. Part (ii) demanded that the letter be interpreted as a generalised number, and the presence of

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question 13(i), along with, in some cases, the encouragement of the interviewer, supported pupils in considering particular values as substitutes for the letter x.

Our concern in this paper is to consider the evidence that we have from our interviews about pupils' ability to use two kinds of reasoning – the 'letter as object' reasoning and the 'letter as generalised number' reasoning described by Küchemann. The former we expect to be characterised by reification ('the x') and/or reference to moving the letter ('put the a and the $3a$ together'). The latter we expect to recognise by reference to operations on the value ('divide $3x$ by 3') or by substitution of particular values into the algebraic expressions.

METHOD

Twenty four pupils from two local comprehensive schools were interviewed in pairs in the first cohort. The pupils were selected by their teachers from those who were willing to take part, with the aim of making compatible pairs of similar attainment. The pairs were distributed across the perceived attainment range in the year group, and contained a balance of boys and girls.

The interviews were conducted by a researcher (the third named author), who presented the questions in written form, and also read them aloud. All interviews were video taped, and audio taped for transcription. Copies of any pupil writing were collected, and added to the transcripts, which were also annotated to include non-verbal behaviour observed in the video tapes. In the analysis presented in this paper we have not generally tried to separate out the contributions of the two individuals to the interview, except where there was an obvious difference of approach.

FORMS OF REASONING

<u>Question 3</u>

Ten of the twelve pairs gave a correct answer to this question, most of them doing so quickly and without very much discussion. The other two pairs tried to answer the question by giving a value to x. One of these pairs, Mollie and Grace (low attainers), gave an answer of 50 and when asked to explain said

> "For x centimetres that's 15, and then we added 15, 15 and 15, which is 45, and then we did 5 add 5, which makes 50".

The other, Nathan and William (high attainers), debated whether x should be taken as 10 centimetres or 15 centimetres and came to no conclusion.

Of the ten pairs who answered correctly, a number gave indications of 'letter as object' reasoning, for example:

> "There's three xs up there and you add them up together. Cause x isn't a number and you can't add the x all together" (Sophie and Lauren, middle attainers)

> "There's three xs so you'd say three x" (Natasha and Holly, middle attainers)

Others gave indications of interpreting x as a number, for example:

> "Cause there's three sides that says x centimetres, so it's $3x$ centimetres" (Kieran and Emily, high attainers)

"There's three xs so it would be three times x" (Adam and Connor, middle attainers)

Question 12

The first two parts of question 12 were answered correctly by all pairs and presented little difficulty for most.

One pair gave a very clear indication of using 'letter as object' reasoning:

> "anything can be an a, so you can put two apples and five apples, so then you can, like, add the five and the two, which is seven, and then you can put seven a equals seven apples" (Mollie and Grace, low attainers)

Others were less clear:

> "there's two a there and an a there and they're exactly the same, well, not in number wise, but they're both as, so you can add those two, which is three a" (Olivia and Lucy, low attainers)

Question 12(iii) proved more challenging. Three pairs failed to reach the correct answer, whilst a fourth pair did not agree on their answer. One of the three pairs that failed to reach an answer explained their difficulties as follows:

> "No, because it's three a minus b and you can't, normally like this when you can put the a with that one, but you can't put the a with the three a because three a, that a is being added to the answer that you get from three a minus b" (Amy and Georgia, high attainers)

Of the successful pairs, few gave any explanation of their answers, but one pair gave some insight into their reasoning:

> "It would be, say, forget about the take away b for the moment, we'll put three a add the other a, so that would be four a, take away the b" (Mollie and Grace, low attainers)

Question 12(iv) caused difficulties for nearly all the pairs. Four pairs achieved some degree of success, and these all involved some substitution in their reasoning.

Kieran and Emily (high attainers) disagreed at the outset, Emily interpreting the bracket as a signal to multiply, apparently because they had been expanding brackets recently in lessons. Kieran began by saying

> "I think it would just be plain a. Cause it's a minus b, but there's a plus b as well, so the b, both bs would cancel out"

A discussion ensued in which Emily, encouraged by the interviewer, substituted a equals 2 and b equals 1 into the expression, but failed to see the significance of the result. Kieran responded with

> "If you don't get it in algebra, why don't you just change it to numbers and then do it in numbers and change it back into algebra and then you've got the answer, like you just did then".

Emily remained unconvinced.

Adam and Connor (middle attainment) had already used substitution in 12(i) to establish that the expression should be simplified to $7a$ and not $7a^2$. For part (iv) Connor was at first interested in whether a is bigger than b, and said:

> "So if a is bigger it would stay the same, because it's take away b and then plus b"

In order to address what happens when a is smaller, they both choose more values to substitute and become convinced that, whatever the values of a and b, the expression is equivalent to a.

Amy and Georgia (high attainers) disagreed at first. Georgia suggested "*a* plus *b*?" and Amy responded, "No, I'd say *a* on its own would be that".

A discussion followed in which Georgia became persuaded of Amy's view. During this Amy wrote:

$a = \cancel{2}5 \quad b = \cancel{8}3$

$5 - 3 + 3$

Mollie and Grace (low attainers), having acknowledged the meaning of the brackets, said:

> "So you'd say *a* take away *b*, um, they can be any numbers, so it doesn't matter what numbers they are"

They went on to collaborate over the substitution ($a = 2$, quickly changed to $a = 4$, $b = 3$) as follows:

Grace	So it could be two
Molly	Take away three. No.
Grace	No, four
Molly	~ yeh. No. Four take away three, so, and then
Grace	The answer could be one
Molly	One, and then you add b
Grace	Add b
Molly	Which could be, hold on, yeh, three, then it would be, what, um
Grace	Or you could do four take away three
Molly	Three, is one. And then because the *b* is a three, so it would be four, so that's four

They do not reach a final statement that the expression is equivalent to *a*, but it is not clear whether this is because they do not see that this is the case or because they are encouraged to go on to the next question.

Of the other eight pairs several mentioned the difficulty created by the presence of brackets in the question, and $a + 2b$ and $a - 2b$ were popular answers.

<u>Question 13</u>

All the pairs were successful with 13(i). Some pairs explicitly calculated before announcing a decision, whilst others announced that it was true without obviously calculating. Some of these made reference to a calculation when asked for a justification.

Ten pairs were agreed that the statement in 13(ii) was true, whilst two agreed it was false. Of these ten, six pairs did not make use of any particular values in justifying the truth of the statement. Typical responses were:

> "cause you've got a number, you times three, then you're dividing it by three, so it's the same" (Amy and Georgia, high attainers)
>
> "yeah, that's the same, because there's three of them, and so divide by three would just take you back to *x* and that would be *x*" (Adam and Connor, middle attainers)

A further three of the successful pairs came to their conclusion having been prompted by the interviewer to consider particular values for *x*. For example, Natasha and Holly (middle attainers) worked as follows:

Holly	But it probably will be true then. Um, because I think that, If you do, if you do six; six plus six plus six is eighteen, then you have to do eighteen divided by three which is six
Interviewer	uh-huh
Natasha	and you do ten as well, it would be thirty, divided by three is ten, and then
Holly	Yeh
Natasha	Seven, twenty one divided by three is seven
Holly	Yeh, it's true.

The final successful pair, Mollie and Grace (low attainers), arrive at a similar conclusion after spontaneously deciding to try 'any numbers', though with considerably more hesitation over the calculations required.

The two pairs who agreed that the statement was false made no attempts to substitute particular values and seemed confused by the syntax.

DISCUSSION

After a year or more of algebra instruction these pupils showed a good deal of competence with basic 'letter as object' reasoning, not only in simple cases of questions 3 and 12(i), (ii), but also in the more demanding context of 12(iii). Question 12(iv) proved very challenging, even for high attainers. Two features of this question seem to have caused difficulty, i.e. the brackets and the negative sign. Difficulties with negative signs have already been documented, and the literature is well summarised in Vlassis (2002), which also sheds some new light on possible reasons for the difficulties. Our interest here is in the strategies used by the pupils who were successful in 12(iv) which was a difficult and non-routine problem. The use of substitution was part of this successful strategy in each case, and the successful students came from across the attainment range.

Similarly, in question 13 we observed some pupils working competently with a 'letter as object' approach, and others addressing this non-routine problem successfully using substitution.

Bazzini et al (2001) use Frege's model of sign/sense/denotation to describe pupils' engagement with algebraic symbolism. They describe how for some students the connection between sign, sense and denotation is lost as they follow a procedure, whereas for others different senses of the same sign can be activated simultaneously so that non-routine problems can be approached. The appropriate use of substitution in the problems we set seems to us to be evidence of an activation of a sense for the algebraic expression which is different from a 'letter as object' understanding.

Cerulli and Mariotti (2001) describe the work of Francesca who was able to transform the expression $(a + b)(a - b)$ into $a^2 - b^2$, but was unsure whether $(10 + 13)(25 - 3)$ was the same as $10 \times 25 - 10 \times 3 + 13 \times 25 - 13 \times 3$. For Francesca it seems that in the algebraic context the sense being activated relates to symbols which can be manipulated according to certain rules. There is evidence that she does not recognise a numerical analogue of the expression as being subject to the same rules of manipulation. When presented with a

similar expression containing values rather than algebraic symbols she does not immediately see it as a particular example of a general equivalence. The implication is that she was not seeing $(a + b)(a - b)$ as a generalisation of particulars, but merely as a symbolic expression. In Mason's terms she has not 'seen the particular in the general'.

In our experience of teaching programmes in the UK, the activity of substitution is practised in the context of evaluation of algebraic expressions, but it is not often linked with developing thinking about symbolic manipulation. Indeed substitution may be widely regarded as not part of a scheme of algebraic development (See, for example, Ursini and Trigeros, 2001).

Bazzini et al (2001) recognize the importance of flexibility, or the activation of different senses, in the solution of non-routine problems. Seeing the particular in the general, as evidenced by a purposeful use of substitution, as well as operating at a general symbolic level, offers one form of flexibility. We have evidence that continuing to see the particular in the general can help pupils to 'activate different senses' and solve the non-routine problem.

Within our teaching programme we exploit opportunities to support pupils in understanding the 'particular and general' nature of algebraic symbols. The spreadsheet environment contains a powerful ambiguity: when a formula is entered in a column, it can be 'filled down' to operate not just on a single cell, but on a range of cells in a column. The symbol used as a cell reference can then be seen as both particular (the number I am going to enter in this cell) and general (all the values I may enter in this column). In future interviews we plan to look for evidence of how working in the spreadsheet environment has supported the development of flexibility in pupils' algebraic thinking.

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